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SOLVABLE GROUPS HAVING SYSTEM NORMALIZERS OF PRIME ORDER

BY

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ABSTRACT. Let G be a solvable group having system normalizer D of prime order. If G has all Sylow groups abelian then we prove that $l(G) = l(C_G(D)) + 2$, provided $l(G) \geq 3$ (here $l(H)$ denotes the nilpotent length of the solvable group H). We conjecture that the above result is true without the condition on abelian Sylow subgroups. Other special cases of the conjecture are handled.

Let H be a finite solvable group and A a group of automorphisms of H. There are many results that give bounds for the nilpotent length of H in terms of certain properties of A. In this note we are concerned with the case where A has prime order and A is a system normalizer of the group $G = HA$. That is, we assume that G has no central chief factor contained in H. In this situation there is surprisingly exact information available. We conjecture that $l(G) = l(C_G(A)) + 2$ provided $l(G) \geq 3$ (by $l(K)$ we mean the nilpotent length of the solvable group K). In this paper we prove special cases of the conjecture. For example if G is an A-group (all Sylow subgroups abelian) or if $l(G) = 4$, then the conjecture is true. We remark that these results are best possible in the sense that if either condition $|A|$ prime or A a system normalizer of $G = HA$ is relaxed then examples can be constructed showing that precise equalities on $l(G)$ in terms of $l(C_G(A))$ cannot always be expected.

We call attention to the preliminary Propositions 2 and 3, where we prove results that guarantee the existence of fixed points contained in fixed cosets, in the case of nonrelatively prime action.

Throughout the paper all groups will be solvable. If G is a group, $l(G)$ is the nilpotent length of G. We let $\mathfrak{N}^i$ denote the saturated formation of groups of nilpotent length at most i.

The following results can be used to obtain fixed points in fixed cosets, in the case of nonrelatively prime action. We reduce the situation to Theorem 3.3 of [2].

Proposition 1. *Let P be a cyclic p-group acting fixed-point-free on a group Q. Let PQ act on a p-group R so that Q acts on R without fixed points. If H/K*

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is PQR-invariant and $H \leq R$, then $C_{H/K}(P) = C_H(P)K/K$.

Proof. Suppose $xK \in C_{H/K}(P)$. Then $P^x K = PK$ and $KQP = KQP^x$. Also the chief factors of KQP contained in KQ are eccentric, so P is a system normalizer of KQP. Theorem 3.3 of [2] implies that P, P^x are conjugate in KQP. Thus there exist elements $k \in K$, $q \in Q$ such that $P^x = P^{kq}$. Then $PH/H = P^xH/H = P^{kq}H/H = P^qH/H$ and it follows that $q = 1$. Thus $P^x = P^k$ and $xk^{-1} \in N(P) \cap H \leq C(P)$. This completes the proof of Proposition 1.

Corollary 2. *Let $l(G) = 3$ and suppose that a system normalizer D of G is cyclic of prime order. If H/K is a G-invariant factor of $G^{\mathfrak{N}^2}$, then each coset of K in H centralized by D contains an element centralized by D.*

Proof. Let $|D| = p$. Since $G^{\mathfrak{N}^2}$ is nilpotent, H/K is nilpotent and we may assume that H/K is a p-group. Also we may assume that K is a p-group. Let L be an $\mathfrak{N}^2$-normalizer of G with $D \leq L$ [1, Theorem 4.9]. Then $L = DL_0$ where $L_0 = L \cap G^{\mathfrak{N}}$. Since H is hypereccentric and H is a p-group, the p-complement Q of L_0 acts without fixed points on H. As D acts fixed-point-freely on Q, we can apply Proposition 1.

The following result is similar in nature to Proposition 1. We will not need this result in what follows although it can be used in similar situations. We remark that an alternate proof is possible using a result of Thompson (Theorem 1 of [3]).

Proposition 3. *Let P be a cyclic p-group, $p > 2$, and suppose P acts fixed-point-freely on a group Q. Let PQ act on the class 2 p-group R, and assume $[R, Q] = R$. If H is a PQ-invariant subgroup of $Z(R)$ such that R/H is elementary abelian, then each coset of H in R fixed by P contains an element fixed by P.*

Proof. Let $G = RQP$ be a minimal counterexample. Suppose that P fixes the coset xH. Write $H = H_1 \times H_2$ where $H_1 = C_H(Q)$ and $H_2 = [H, Q]$, and suppose $H_1 > 1$ and $H_2 > 1$. Then from the groups G/H_1 and G/H_2 we find elements x_1, x_2 in xH such that P fixes x_1H_1 and x_2H_2. Consequently there exist elements $h_1 \in H_1$, $h_2 \in H_2$ such that P fixes $x_1H_1 = xh_2H_1$ and $x_2H_2 = xh_1H_2$. Let $P = \langle g \rangle$. There exist elements $t_1 \in H_1$, $t_2 \in H_2$ such that $(xh_2)^g = xh_2t_1$ and $(xh_1)^g = xh_1t_2$. But $xh_1 = xh_2(h_2^{-1}h_1)$, and so $(xh_1)^g = xh_2t_1(h_2^{-1})^g h_1^g = xh_1t_2$. It follows that $h_2(h_2^{-1})^g t_2^{-1} = h_1(h_1^{-1})^g t_1^{-1}$. Since $g \in N(Q)$, g normalizes H_1 and H_2, and consequently $t_2 = h_2(h_2^{-1})^g$ and $(xh_1)^g = xh_1h_2(h_2^{-1})^g$. This gives $(xh_1h_2)^g = xh_1h_2$. Therefore $H_1 = 1$ or $H_2 = 1$.

If $H_1 = 1$, then since R/H is abelian and $[R, Q] = R$, we have Q acting on R without fixed points. In this case Proposition 1 gives a fixed point of P in xH.

Thus we have $H_2 = 1$ and $H \leq Z(RQ)$. We next show that $H = Z(R)$. If $H < Z(R)$ then write $Z(R) = H \times L$ where $L = [Z(R), Q]$. From the group G/L we find an element $h \in H$ such that xhL is fixed by P. Then $(xh)^g = xhl$ for some $l \in L$, and $x^g = xh_1$ for some $h_1 \in H$. It follows that $l = 1$ and g fixes xh. Thus $H = Z(R)$ as claimed. Suppose $[H, P] \neq 1$. Then considering the series $H > [H, P] > [H, P, P] > \cdots > 1$ we find an element of the form $1 \neq z = [g, z_1]$ where z is centralized by P and $z_1 \in H$. Then $\langle z \rangle \leq Z(G)$ and from $G/\langle z \rangle$ we have an element $h \in H$ such that P fixes $xh\langle z \rangle$. Then $(xh)^g = xhz^i$ for some i, $(xh)^g = xh[g, z_1^i]$, and $(xhz_1^i)^g = xhz_1^i$. Therefore we must have $H = Z(G)$.

If $H = Z(R)$ is not cyclic then letting N_1, N_2 be proper subgroups of $Z(R)$ with $N_1 \cap N_2 = 1$ we get a contradiction by considering G/N_1 and G/N_2. Therefore $Z(R)$ is cyclic and since $[Q, R] = R$, R is extraspacial. Thus, $|R\colon \Omega_1(R)| \leq p$ and hence $R = \Omega_1(R)$. Now let A be the subgroup of the automorphism group of R that centralizes $Z(R)$. Then $PQ \leq A$ and A is the semidirect product of $\mathrm{Inn}(R)$ with $L \cong \mathrm{Sp}(V)$, where V is the vector space $R/Z(R)$. This may be checked as follows. First show that $\mathrm{Inn}(R)$ is the group of automorphisms of R centralizing $R/Z(R)$. Then for each $g \in \mathrm{Sp}(V)$ it is possible to produce an automorphism of R having the appropriate action on $R/Z(R)$ and centralizing R. This proves that $A/\mathrm{Inn}(R) \cong \mathrm{Sp}(V)$ and the only question is whether the extension splits. Since p is odd, $\mathrm{Sp}(V)$ has a central involution, and consequently A contains an involution t such that t inverts $\mathrm{Inn}(R)$. Then $L = C_A(t) \cong \mathrm{Sp}(V)$.

Now $\mathrm{Inn}(R)$ is L-isomorphic to V. Since $(|Q|, p) = 1$ we conjugate if necessary to assure $Q \leq L$. $[Q, R] = R$ implies that $[Q, \mathrm{Inn}\, R] = \mathrm{Inn}\, R$ and it follows that $N_A(Q) \leq L$. In particular $PQ \leq L$ and PQ centralizes the central involution t of L. Now $\langle x, Z(R)\rangle = Z(R) \times R_0$ where t centralizes $Z(R)$ and inverts R_0. P centralizes t and hence normalizes R_0. However P centralizes $\langle x, Z(R)\rangle/Z(R)$ and R_0 contains precisely one element of $xZ(R)$. Therefore P centralizes R_0 and an element in $xZ(R)$. This completes the proof of Proposition 3.

The next lemma is trivial but useful.

Lemma 4. *Let $l(G) = n$ and $H \leq G$. If H has a chief factor M/N such that $l(H/C_H(M/N)) = n - 1$, then $l(H) = n$.*

Proof. $l(H) \leq n$ and $\mathrm{Fit}(H)$ centralizes each chief factor of H.

Theorem 5. *Let G be a finite solvable group having system normalizer D of prime order. If $l(G) = 4$, then $l(C_G(D)) = 2$.*

Proof. Suppose the result false and let G be a counterexample of minimal order. Write $G = DG^{\mathfrak{N}}$, where $G^{\mathfrak{N}}$ is the nilpotent residual of G and where $|D| = p$. Then $D \cap G^{\mathfrak{N}} = 1$, $G^{\mathfrak{N}}$ is hypereccentric, and $C_G(D) = D \times C_{G^{\mathfrak{N}}}(D)$. Also D acts fixed-point-freely on $G^{\mathfrak{N}}/G^{\mathfrak{N}^2}$, so that $C_{G^{\mathfrak{N}}}(D) \leq G^{\mathfrak{N}^2}$ and $l(C_G(D)) =$

$l(C_{G^{\mathfrak{N}}}(D)) \leq l(G^{\mathfrak{N}^2}) = 2$. As G is a counterexample we have $C_{G^{\mathfrak{N}}}(D)$ nilpotent. Let N be a minimal normal subgroup of G, $|N| = q^a$, q prime.

We claim that $N = G^{\mathfrak{N}^3} = \mathrm{Fit}(G)$ and N is the unique minimal normal subgroup of G. It suffices to prove the equalities, and we first show that $N = G^{\mathfrak{N}^3}$. Suppose that $q \neq p$. Then $C_{G/N}(DN/N) = C(D)N/N$. Thus if $G^{\mathfrak{N}^3} \not\leq N$, then $C(D)N/N$ has length 2, contradicting the choice of G. Thus $N = G^{\mathfrak{N}^3}$. Now suppose $q = p$. Since D is fixed-point-free on $G^{\mathfrak{N}}/G^{\mathfrak{N}^2}$, $N \leq G^{\mathfrak{N}^2}$. Let T be an $\mathfrak{N}^3$-normalizer of G containing D [1, Theorem 4.9] and consider the group $G_0 = NT$. Since $G^{\mathfrak{N}^3}$ is nilpotent and $G = G^{\mathfrak{N}^3}T$, it follows that N is a minimal normal subgroup of G_0 and by the covering properties of T, either $N \leq T$ or $N \cap T = 1$. If $l(G_0) = 4$, then since D is a system normalizer of G_0, we conclude from the minimality of G that $G = G_0$ and $N = G^{\mathfrak{N}^3}$. Now suppose $l(G_0) = 3$. Then $N \leq T$ and $G_0 = T$ has nilpotent length 3. Also $l(G/C_G(N)) = 2$. Indeed, $G = G^{\mathfrak{N}^3}T$ and both $G^{\mathfrak{N}^3} \leq \mathrm{Fit}(G)$ and $\mathrm{Fit}(T)$ centralize N. This shows $l(G/C_G(N)) \leq 2$. If $l(G/C_G(N)) = 1$, then $G^{\mathfrak{N}} \leq C_G(N)$, and since N is a p-group, $N \leq Z(G)$, a contradiction. In particular $N \leq Z(G^{\mathfrak{N}^2})$. Let $C = C_{G^{\mathfrak{N}}}(DN/N)$ and suppose $G^{\mathfrak{N}^3} \not\leq N$. The minimality of G implies $l(C/N) = 2$. If $G^{\mathfrak{N}^3}$ is not a p-group, let M be a minimal normal subgroup of G, $M \leq O_{p'}(G^{\mathfrak{N}^3})$. By the above it follows that $M = G^{\mathfrak{N}^3}$. Since $l(C/N) = 2$ and since $C \leq G^{\mathfrak{N}^2}$, some element xN of C/N of order prime to $|M|$ does not centralize $C_M(D)$. Otherwise $C_M(D)N/M$ would be hypercentral in C/N and since $C/C_M(D)N$ is nilpotent, this would contradict $l(C/N) = 2$. Since $G/N = (T/N)(MN/N)$ and since $(|xN|, |M|) = 1$, we may assume that $x \in T$. Corollary 2 implies that there is a $y \in C(D)$ such that $yN = xN$. As $N \leq C(M)$, we have $C_M(D)\langle y\rangle$ nonnilpotent and contained in $C(D)$. Thus we may assume that $G^{\mathfrak{N}^3}$ is a p-group. Let $g \in C \leq G^{\mathfrak{N}^2}$ and $h \in D$. Then $C_C(h)^g = C_C(h^g) = C_C(hn)$ for some $n \in N$. Since $N \leq Z(G^{\mathfrak{N}^2})$, this gives $C_C(h^g) = C_C(h)$ and $C_C(D) \trianglelefteq C$. Also all p'-elements of C are contained in $C_C(D)$. Indeed if x is a p'-element of C, then $x \in C(DN/N)$ and $x \in C(N)$ implies $x \in C(DN)$. Since $C(D)$ is nilpotent, C contains a normal nilpotent p-complement. On the other hand $G^{\mathfrak{N}^3}$ is a p-group and $C \leq G^{\mathfrak{N}^2}$. Thus C also contains a normal Sylow p-group. This implies that C is nilpotent, whereas $l(C/N) = 2$. This contradiction shows that $N = G^{\mathfrak{N}^3}$ in all cases. In particular N is unique. With T as above, we have $\mathrm{Fit}(G) = N(\mathrm{Fit}(G) \cap T)$. Since N is minimal normal in G, $\mathrm{Fit}(G) \cap T$ centralizes N and $\mathrm{Fit}(G) \cap T \trianglelefteq G$. The uniqueness of N shows that $\mathrm{Fit}(G) \cap T = 1$. This proves the claim.

We next claim that G can be expressed $G = NRQD$ where $D \leq N(Q)$, Q minimal normal in QD, $QD \leq N(R)$, and R is a minimal normal subgroup of RQD. To see this first write $G = NT$ where T is an $\mathfrak{N}^3$-normalizer of G and $D \leq T$. Let L be an $\mathfrak{N}^2$-normalizer of T with $D \leq L$. Then $L = DW$ with $W = L \cap G^{\mathfrak{N}}$. Let $1 \neq R$

be the Sylow r-subgroup of $T^{\mathfrak{N}^2}$. Since $T^{\mathfrak{N}^2}$ is nilpotent, R is a direct factor of $T^{\mathfrak{N}^2}$ and $T^{\mathfrak{N}}$ centralizes no G-chief factor of $R/\Phi(R)$. But a G-chief factor of $R/\Phi(R)$ is a T-chief factor of $R/\Phi(R)$, and hence an L-composition factor of $R/\Phi(R)$. Thus Lemma 4 implies $l(RL) = l(RWD) = 3$. Now $NR \trianglelefteq G$, $C_G(N) = N$, and $N = \mathrm{Fit}(G)$. Thus N is R-hypereccentric. Considering the group $NRWD$ we apply Lemma 4 and obtain $l(NRWD) = 4$. The minimality of G gives $G = NRWD$. Suppose that R_0 is a minimal normal subgroup of RWD with $R_0 < R$.

Suppose $NR_0WD = NRWD$. Then $R_0WD = T = RWD$ and $R = R_0(W \cap R)$. However $R/\Phi(R)$ is $\mathfrak{N}^2$-hypereccentric, so $W \cap R \leq \Phi(R)$. Consequently $R = R_0\Phi(R)$, contradicting $R_0 < R$. Thus $NR_0WD < NRWD$. The minimality of G implies $l(NR_0WD) = 3$ ($l(NR_0WD) \leq 2$ implies $N \leq L = WD$, whereas $N \cap WD \leq N \cap T = 1$). Then $(R_0WD)^{\mathfrak{N}^2}$ is trivial on each NR_0WD-chief factor contained in N. However $(R_0WD)^{\mathfrak{N}^2} \leq R_0$ and $(|R_0|, |N|) = 1$. Thus $(R_0WD)^{\mathfrak{N}^2} \leq C(N) \cap R_0 = N \cap R_0 = 1$, and $l(R_0WD) = 2$. Now R_0 is minimal normal in R_0WD and $R_0WD/C(R_0) \cap R_0WD$ is nilpotent. Thus $W \leq C(R_0)$ and D acts fixed-point-freely on R_0 (otherwise R_0N/N would be a central factor in G). We are assuming that $C_G(D)$ is nilpotent. Since N is the unique minimal normal subgroup of G and since $G = NT$ with $N \cap T = 1$, we have $N = C_G(N)$ and $(|N|, |R|) = 1$. Our assumption that $C_G(D)$ is nilpotent is equivalent to $C_R(D) \leq C(C_N(D))$.

Let K be a finite splitting field for all subgroups of G, $\mathrm{char}(K) = q$. Then $K \otimes N = N_1 \oplus \cdots \oplus N_t$, a direct sum of absolutely irreducible, algebraically conjugate representations of RWD. Also RWD acts faithfully on each N_i. Now $C_{K\otimes N}(D) = K \otimes C_N(D) = C_{N_1}(D) \oplus \cdots \oplus C_{N_t}(D)$ and $C_{K\otimes N}(C_R(D)) = K \otimes C_N(C_R(D)) = C_{N_1}(C_R(D)) \oplus \cdots \oplus C_{N_t}(C_R(D))$. Thus letting $V = N_1$ we have $C_R(D) \leq C(C_V(D))$. Apply Clifford's theorem to R_0. $V_{R_0} = V_1 \oplus \cdots \oplus V_n$, a decomposition into homogeneous components. On each V_i, R_0 induces a group of scalar transformations. As D is fixed-point-free on R_0, D acts semiregularly on the V_i. Thus if $0 \neq x_i \in V_i$ then $x = \sum_{g\in D} x_i^g$ is nonzero and centralized by D. Thus, $C_R(D)$ centralizes x. However $R_0 \leq Z(R)$ implies that R stabilizes each V_i and so $C_R(D)$ stablizes each V_i. Consequently $C_R(D)$ centralizes x_i, and as i and $x_i \in V_i$ were arbitrary, $C_R(D) \leq C(V) = 1$. But this implies D is fixed-point-free on RW, whereas this group is not nilpotent. This is a contradiction and so R is minimal normal in RWD.

Now let Q be a minimal normal subgroup in DW. As W is nilpotent, D is fixed-point-free on Q. Consider the group $G_0 = NRQD$. Then the minimality of G yields $G = G_0$ if $l(G_0) = 4$. Suppose $l(G_0) = 3$. Since D is fixed-point-free on Q and since R acts without fixed points on N, the only possibility is that $Q \leq C(R)$. Also $l(RWD) = 3$ and R minimal normal in RWD implies that $R \cap WD = 1$.

As above write $K \otimes N = N_1 \oplus \cdots \oplus N_t$, set $V = N_1$ and consider $V_Q = V_1 \oplus \cdots \oplus V_b$. D permutes the V_i semiregularly and $R \le C(Q)$ fixes each V_i. Once again we get the contradiction $C_R(D) \le C(V) = 1$. This completes the proof of the claim.

We now complete the proof of Theorem 5. We have $C_R(D) \le C(C_N(D))$. As above choose a finite field K of characteristic q such that K is a splitting field for all subgroups of G, and obtain a faithful irreducible $K(RQD)$-module V satisfying $C_R(D) \le C(C_V(D))$. Write $V_R = V_1 \oplus \cdots \oplus V_k$, where the V_i are the homogeneous components. Then QD is transitive on $V_1, \cdots, V_k$. If $k = 1$, then R is cyclic, which is impossible since QD acts faithfully on R. Thus $k > 1$.

If D is semiregular on $V_1, \cdots, V_k$, then as in the above claim $C_R(D) \le C(V) = 1$, a contradiction. Thus we may assume that D fixes V_1. Since $C_{QD}(D)$ is transitive on the V_i's fixed by D, D fixes only V_1. Now D is semiregular on $V_2, \cdots, V_k$ and it follows that $C_R(D) \le C(V_2 \oplus \cdots \oplus V_k)$. Thus $C_R(D)$ acts faithfully on V_1 and $C_R(D)$ is cyclic. Say $C_R(D) = \langle t \rangle$. The stabilizer of V_1 in RQD acts irreducibly on V_1, and this stabilizer is just RD. Now $R/\ker V_1$ is represented as scalar matrices and consequently $R/\ker V_1 = C_R(D)\ker V_1/\ker V_1$. As RD acts irreducibly and $RD/\ker V_1$ is abelian, we have $\dim(V_1) = 1$. But then the matrix of t has the form

$$\begin{pmatrix} \alpha & & & & \\ & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & 1 \end{pmatrix}$$

and $\det(t) = \alpha \ne 1$. As $t \in R \le (RQD)'$, this is impossible and the proof of Theorem 5 is complete.

Theorem 6. *Let $n \ge 3$ and $G = A_n A_{n-1} \cdots A_1$ be a repeated semidirect product of the abelian groups A_i. Suppose that A_1 is a system normalizer of G, A_1 has prime order p and $(p, |A_i|) = 1$ for $i > 1$. If $l(G) = n$, then $l(C_G(A_1)) = n - 2$.*

Proof. Let G be a minimal counterexample. By Theorem 5 $n > 4$. Write $A_1 = D$. If $\mathfrak{F} = \mathfrak{N}^{n-1}$, then $1 < G^{\mathfrak{F}} \le A_n$ and $G^{\mathfrak{F}}$ is $\mathfrak{F}$-hypereccentric [1, Theorem 5.15]. Let N_n be a minimal normal subgroup of G with $N_n \le G^{\mathfrak{F}}$. N_n is $\mathfrak{N}^{n-1}$-eccentric so that $G/C_G(N_n)$ has length $n - 1$. As $A_n \le C(N_n)$, setting $G_0 = N_n A_{n-1} \cdots A_1$ we have N_n minimal normal in G_0 and $G_0/C_{G_0}(N_n)$ of length $n - 1$. Thus $l(G_0) = n$ by Lemma 4, and the minimality of G implies that $G = G_0$. Also the minimality of G implies that $C_G(N_n) = N_n$. Now let N_{n-1} be a minimal normal subgroup of $A_{n-1} \cdots A_1$ contained in $(A_{n-1} \cdots A_1)^{\mathfrak{N}^{n-2}}$. Then the minimality of G implies that $G = N_n N_{n-1} A_{n-2} \cdots A_1$ and $C_G(N_{n-1}) = N_{n-1}$. Continuing in this way we obtain $G = N_n \cdots N_1$ where for each $i = 1, \cdots, n$, N_i is self-centralizing and minimal normal in $N_i \cdots N_1$.

For each i, let $M_i = N_i \cap C(D)$. Then $M_2 = 1$. As $D = A_1$ is a Sylow p-subgroup of G, $C_G(D) = M_n \cdots M_3 \times D$. We are assuming that $l(M_n \cdots M_1) \leq n-3$. Now $N_{n-1} \cdots N_1 < G$, so that $l(M_{n-1} \cdots M_1) = n-3$. Thus $l(M_n \cdots M_1) = n-3$ and M_n is $\mathfrak{N}^{n-3}$-hypercentral. As $(|M_n|, |M_{n-1}|) = 1$ we have $(M_{n-1} \cdots M_1)^{\mathfrak{N}^{n-4}} \leq C(M_n)$. Also $[M_3, M_4] = (M_3 M_4)^{\mathfrak{N}}$, $[M_3, M_4, M_5] = (M_3 M_4 M_5)^{\mathfrak{N}^2}, \cdots, [M_3, \cdots, M_{n-1}] = (M_{n-1} \cdots M_1)^{\mathfrak{N}^{n-4}}$. Write $N_3 = M_3 \times L_3$ where $L_3 = [D, N_3]$.

Suppose that $[[N_4, M_3], L_3] \neq 1$. Consider the group

$$G_0 = D \cdot L_3 \cdot [N_4, M_3, L_3][N_4, M_3, L_3, N_5][N_4, M_3, L_3, N_5, N_6]$$
$$\cdots [N_4, M_3, L_3, N_5, \cdots, N_n].$$

Then G_0 has the form $G_0 = DL_3 \cdots L_n$ where each $L_i \leq N_i$ and where $1 < L_{i+1} = [L_{i+1}, L_i]$, for $i = 3, \cdots, n-1$. It follows that D is a system normalizer of G_0 and $l(G_0) = n-1$. Thus $l(C_{G_0}(D)) = n-3$. $(C_{G_0}(D))^{\mathfrak{N}^{n-4}} \leq C_{L_n}(D)$ so $C_{L_n}(D)$ is not $\mathfrak{N}^{n-5}$-hypereccentric. In particular $(C_{G_0}(D))^{\mathfrak{N}^{n-5}} \not\leq C(C_{L_n}(D))$. That is $[C_4, C_5, \cdots, C_{n-1}] \not\leq C(C_{L_n}(D))$, where $C_i = C_{L_i}(D)$. Now $C_4 \leq [N_4, M_3, L_3] \leq [N_4, M_3]$. As M_4 is abelian of order prime to $|M_3|$, M_3 centralizes no element of $[N_4, M_3]$. But $C_4 \leq L_4 \cap C(D) \leq N_4 \cap C(D) = M_4$ and M_3 normalizes M_4. Thus $M_4 = [M_4, M_3] \times C_{M_4}(M_3)$ and $[M_4, M_3] = M_4 \cap [N_4, M_3]$. It follows that $C_4 \leq [M_4, M_3]$. Thus $[C_4, \cdots, C_{n-1}] \leq [M_3, M_4, \cdots, M_{n-1}]$, and so $[M_3, \cdots, M_{n-1}] \not\leq C(C_{N_n}(D)) \leq C(C_{L_n}(D))$. As $C(C_{N_n}(D)) = C(M_n)$, this is a contradiction. Thus $[N_4, M_3, L_3] = 1$.

Consider the group $N_4 N_3 N_2 D$. Write $N_2 = S$, $N_3 = Q$, $N_4 = R$, s, q, and r-groups, respectively. We have $Q = Q_0 \times Q_1$ where $Q_0 = C_Q(D)$ and $Q_1 = [Q, D]$. Also R is an irreducible DSQ-module satisfying $[[R, Q_0], Q_1] = 1$. Let K be a finite field of characteristic r such that K is a splitting field for each subgroup of DSQ. Then $K \otimes R = R_1 \oplus \cdots \oplus R_t$ where the R_i are absolutely irreducible, algebraically conjugate DSQ-modules. Since $[K \otimes R, Q_0] = K \otimes [R, Q_0]$ and $C_{K \otimes R}(Q_1) = K \otimes C_R(Q_1)$, it follows that $[R_1, Q_0, Q_1] = 1$. Consider the representation of DSQ on the module $V = R_1$. By Clifford's theorem $V_Q = V_1 \oplus \cdots \oplus V_k$ where the V_i are the homogeneous components of Q and the V_i are permuted transitively by DS. If $k = 1$, then Q is cyclic and $DSQ/C(Q) = DSQ/Q$ is cyclic. This is impossible, and so $k > 1$. Let $V_0 = [C_V(D), Q_0]$. By Theorem 5, $V_0 \neq 0$ and since $V_0 \leq [V, Q_0] \leq C_V(Q_1)$, V_0 is Q-invariant. Thus $V_0 = (V_0 \cap V_1) \oplus \cdots \oplus (V_0 \cap V_k)$. If D is semiregular on $V_1, \cdots, V_k$ then

$V_0 \cap V_i = 0$ for each i, contradicting the fact that $V_0 \neq 0$. Thus D is not semiregular, say D fixes V_1. Then $C_{DS}(D)$ is transitive on the V_i fixed by D, so that D fixes just V_1. In particular $V_0 \leq V_1$. The stabilizer of V_1 is DQ and so DQ acts irreducibly on V.

Let $0 \neq v_i \in V_i$, $i > 1$. Then $v = \Sigma_{g \in D} v_i^g$ is nonzero and centralized by D. So for $t \in Q_0$, we have $v^t - v \in V_0$. But $v^t - v \in V_2 \oplus \cdots \oplus V_k$, so that $v^t = v$ and $v_i^t = v_i$. It follows that Q_0 centralizes $V_2 \oplus \cdots \oplus V_k$. Then Q_0 is faithful on V_1 and since Q induces a cyclic group on V_1, we can write $Q_0 = \langle t \rangle$. Then $DQ/\ker(V_1) = (D \times Q_0)\ker V_1/\ker V_1$ and $\dim V_1 = 1$. Then t is represented on V by the matrix

$$\begin{pmatrix} \alpha & & & & \\ & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & 1 \end{pmatrix}$$

for some $1 \neq \alpha \in K$. This contradicts the fact that $Q_0 \leq Q \leq (DSQ)'$, and Theorem 6 is proved.

One special case of Theorem 6 is a common one in the construction of solvable groups. Namely let $D = A_1$ have prime order p. Let A_2 be a finite faithful irreducible module for D over a field of characteristic prime to p. Now let A_3 be a finite faithful irreducible module for the group A_2A_1 over a field of characteristic prime to p. Continue in this way and let $G = A_n \cdots A_1$. Then as long as $n \geq 3$, $l(G) = l(C(D)) + 2$.

The last result is another special case of Theorem 6.

Theorem 7. *Let G be an A-group having nilpotent length $n \geq 3$ and system normalizer D of prime order. Then $l(C_G(D)) = n - 2$.*

Proof. Suppose $l(G) = n \geq 3$, G is an A-group and D is a system normalizer of prime order. $A_n = G^{\mathfrak{N}^{n-1}}$ is nilpotent and hence abelian. Thus $G = A_nT$ with $A_n \cap T = 1$, for T an $\mathfrak{N}^{n-1}$-normalizer of G [1, Theorem 5.15]. Moreover we may assume $D \leq T$. Inductively we obtain the required factorization, $G = A_n \cdots A_1$. The theorem now follows from Theorem 6.

We conclude the paper by giving examples that show two ways in which Theorem 6 is best possible.

Example 1. Let A be cyclic of order p^3 and let A act irreducibly on a group B in such a way that $C_A(B) = \Omega_2(A)$. Now let BA act irreducibly on a group C so that $C_{BA}(C) = \Omega_1(A)$. Finally let CBA act faithfully and irreducibly on a group D. Then $G = DCBA$ has length 4, system normalizer A, and $l(C_G(A)) = 1$. So the prime order assumption is needed.

Example 2. Let BA be the nonabelian group of order 21, with $|B| = 7$, $|A| = 3$. Then BA acts faithfully on an elementary group C of order 8, in such a way that $C_C(A) = \langle t\rangle$ where t is an involution. Also $C = \langle t\rangle \times [C, A]$. Thus CA has a linear representation Φ over F_3 with kernel $A[C, A]$. Let V be the representation module of Φ^{CBA}. Then V is elementary abelian of order 3^7 and V is faithful. Let $G = VCBA$. Then G satisfies the hypothesis of Theorem 5 and $l(C_G(A)) = 2$.

Now the restriction of V to C splits as the direct sum of the seven distinct nontrivial linear representations of C over F_3 and so $V = V_0 \times V_1$ where $V_0 = [V, t]$, $V_1 = C_V(t)$, $|V_0| = 3^4$, $|V_1| = 3^3$. A centralizes t and so acts on V_0. For $v \in V_0$, $\langle v, v^x, v^{x^2}\rangle$ is A-invariant, where $\langle x\rangle = A$. Thus, under the action of A, V_0 splits as a sum of at least two indecomposable submodules, one of which has dimension 1 or 2. This particular submodule contains an element g such that $(xg)^3 = 1$ and $g \notin [V, A]$. Let $A' = \langle xg\rangle$. Then $VCBA' = G$ and $VCB \cap A' = 1$. However we will show that $C_G(A')$ is nilpotent. To see this first note that $C_G(A')V/V \leq C_G(A'V/V) = C_G(AV/V) = A\langle t\rangle V/V$. Thus if $C_G(A') \not\leq AV$, it must be that A' centralizes some element of tV. Suppose xg centralizes tv, $v \in V$. Then $tv = (tv)^{xg} = (tv^x)^g = t^g v^x = tg^2 v^x$. This gives $g = v^{-1}v^x = [v, x]$, a contradiction. This example shows that it is essential that the automorphism of prime order actually normalizes a Sylow system.

Finally, we remark that it may be possible to weaken the conditions on A, only assuming that A is contained in a system normalizer of G. That is in Theorem 5 we would assume that $|G: G^{\mathfrak{N}}| = p$, $|A| = p$, and A is contained in a system normalizer. This would allow for central factors contained in $G^{\mathfrak{N}}$.

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